Exercise 3

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = 6x + 2x^3 - \int_0^x tu(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 6x + 2x^3 - \int_0^x t \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 6x + 2x^3 - \int_0^x t[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{6x + 2x^3}_{u_0(x)} + \underbrace{\int_0^x [-tu_0(t)]\,dt}_{u_1(x)} + \underbrace{\int_0^x [-tu_1(t)]\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}
u_0(x) &= 6x + 2x^3 \\
u_1(x) &= \int_0^x [-tu_0(t)]\,dt = -\int_0^x t(6t + 2t^3)\,dt = -2x^3 - \frac{2}{5}x^5 \\
&\vdots
\end{aligned}$$

The noise terms $\pm 2x^3$ appear in both $u_0(x)$ and $u_1(x)$. Cancelling $2x^3$ from $u_0(x)$ leaves $6x$. Now we check to see whether $u(x) = 6x$ satisfies the integral equation.

$$\begin{aligned}
6x &\stackrel{?}{=} 6x + 2x^3 - \int_0^x 6t^2\,dt \\
6x &\stackrel{?}{=} 6x + 2x^3 - 2x^3 \\
6x &= 6x
\end{aligned}$$

Therefore,

$$u(x) = 6x.$$